31. PROBABILITY DISTRIBUTION

RANDOM VARIABLE *Let S be the sample space associated with a given random experiment. A real-valued function X which assigns a unique real number $X(w)$ to each $w \in S$, is called a random variable.*

A random variable which can assume only a finite number of values is called a *discrete random variable*.

Example Suppose that a coin is tossed twice.

Then, sample space $S = \{TT, HT, TH, HH\}$.

Consider a real-valued function X on S, defined by

$X : S \to R : X(w) =$ number of heads in w, for all $w \in S$.

Then, X is a random variable such that

$X(TT) = 0, X(HT) = 1, X(TH) = 1$ and $X(HH) = 2$.

Clearly, range $(X) = \{0, 1, 2\}$.

Probability Distribution of a Random Variable

A description giving the values of a random variable along with the corresponding probabilities is called the probability distribution of the random variable.

If a random variable X takes the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ then the probability distribution of X is given by

X	x_1	x_2	x_3	$\ldots$	$\ldots$	x_n
$P(X)$	p_1	p_2	p_3	$\ldots$	$\ldots$	p_n

REMARK The above probability distribution of X is defined only when

(i) each $p_i \geq 0$ (ii) $\sum_{i=1}^{n} p_i = 1$

Mean and Variance of Random Variables

Let a random variable X assume values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively such that each $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Then *mean* of X, denoted by μ [or expected value of X, denoted by $E(X)$], is defined as

$$\mu = E(X) = \sum_{i=1}^{n} x_i \, p_i .$$

And, the *variance*, denoted by σ^2, is defined as

$$\sigma^2 = (\Sigma x_i^2 p_i - \mu^2).$$

Standard deviation, σ, is given by

$$\sigma = \sqrt{\text{variance}}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the mean, variance and standard deviation of the number of tails in two tosses of a coin.*

SOLUTION In two tosses of a coin, the sample space is given by

$$S = \{HH, HT, TH, TT\}.$$

$\therefore \quad n(S) = 4.$

So, every single outcome has a probability $\frac{1}{4}$.

Let X = number of tails in two tosses.

In two tosses, we may have no tail, 1 tail or 2 tails.

So, the possible values of X are 0, 1, 2.

$$P(X = 0) = P(\text{getting no tail}) = P(HH) = \frac{1}{4}.$$

$$P(X = 1) = P(\text{getting 1 tail}) = P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}.$$

$$P(X = 2) = P(\text{getting 2 tails}) = P(TT) = \frac{1}{4}.$$

Hence, the probability distribution of X is given by

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$\therefore \quad \text{mean, } \mu = \Sigma x_i\, p_i = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right) = 1.$$

$$\text{Variance, } \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left[\left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(4 \times \frac{1}{4}\right)\right] - 1^2$$

$$= \frac{1}{2}.$$

$$\text{Standard deviation, } \sigma = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414}{2} = 0.707.$$

EXAMPLE 2 *Find the mean, variance and standard deviation of the number of heads when three coins are tossed.*

SOLUTION Here, $S = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$.

$\therefore \quad n(S) = 8.$

So, every single outcome has a probability $\frac{1}{8}$.

Let X = number of heads in tossing three coins.

The number of heads may be 0, 1, 2, or 3.

So, the possible values of X are 0, 1, 2, 3.

$P(X = 0) = P(\text{getting no head}) = P(TTT) = \frac{1}{8}.$

$P(X = 1) = P(\text{getting 1 head}) = P(TTH \text{ or } THT \text{ or } HTT) = \frac{3}{8}.$

$P(X = 2) = P(\text{getting 2 heads}) = P(THH, HTH, HHT) = \frac{3}{8}.$

$P(X = 3) = P(\text{getting 3 heads}) = P(HHH) = \frac{1}{8}.$

Thus, we have the following probability distribution:

$X = x_i$	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\therefore \quad \text{mean}, \mu = \Sigma x_i\, p_i = \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = \frac{3}{2}.$

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left[\left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(4 \times \frac{3}{8}\right) + \left(9 \times \frac{1}{8}\right) - \frac{9}{4}\right] = \frac{3}{4}.$$

Standard deviation, $\sigma = \frac{\sqrt{3}}{2}.$

EXAMPLE 3 *A die is tossed once. If the random variable X is defined as*

$$X = \begin{cases} 1, \text{ if the die results in an even number} \\ 0, \text{ if the die results in an odd number} \end{cases}$$

then find the mean and variance of X.

SOLUTION In tossing a die once, the sample space is given by

$$S = \{1, 2, 3, 4, 5, 6\}.$$

$\therefore \quad P(\text{getting an even number}) = \frac{3}{6} = \frac{1}{2},$

$P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}.$

As given, X takes the value 0 or 1.

$$P(X=0) = P(\text{getting an odd number}) = \frac{1}{2}.$$

$$P(X=1) = P(\text{getting an even number}) = \frac{1}{2}.$$

Thus, the probability distribution of X is given by

$X = x_i$	0	1
p_i	$\frac{1}{2}$	$\frac{1}{2}$

$$\therefore \quad \text{mean}, \mu = \Sigma x_i\, p_i = \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right) = \frac{1}{2}.$$

$$\text{Variance}, \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}.$$

EXAMPLE 4 *Find the mean, variance and standard deviation of the number of sixes in two tosses of a die.*

SOLUTION In a single toss, we have

$$\text{probability of getting a six} = \frac{1}{6}, \text{ and}$$

$$\text{probability of getting a non-six} = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Let X denote the number of sixes in two tosses.

Then, clearly X can assume the value 0, 1, or 2.

$P(X=0) = P$[(non-six in the 1st draw) and (non-six in the 2nd draw)]

$= P(\text{non-six in the 1st draw}) \times P(\text{non-six in the 2nd draw})$

$$= \left(\frac{5}{6} \times \frac{5}{6}\right) = \frac{25}{36}.$$

$P(X=1) = P$[(six in the 1st draw and non-six in the 2nd draw) or(non-six in the 1st draw and six in the 2nd draw)]

$= P(\text{six in the 1st draw and non-six in the 2nd draw})$

$+ P(\text{non-six in the 1st draw and six in the 2nd draw})$

$$= \left(\frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6}\right) = \left(\frac{5}{36} + \frac{5}{36}\right) = \frac{10}{36} = \frac{5}{18}.$$

$P(X=2) = P$[six in the 1st draw and six in the 2nd draw]

$= P(\text{six in the 1st draw}) \times P(\text{six in the 2nd draw})$

$$= \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{36}.$$

Hence, the probability distribution is given by

$X = x_i$	0	1	2
p_i	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

$$\therefore \quad \text{mean}, \mu = \Sigma x_i\ p_i = \left(0 \times \frac{25}{36}\right) + \left(1 \times \frac{5}{18}\right) + \left(2 \times \frac{1}{36}\right) = \frac{6}{18} = \frac{1}{3}.$$

$$\text{Variance}, \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left[\left(0 \times \frac{25}{36}\right) + \left(1 \times \frac{5}{18}\right) + \left(4 \times \frac{1}{36}\right) - \frac{1}{9}\right] = \frac{5}{18}.$$

$$\text{Standard deviation}, \sigma = \sqrt{\frac{5}{18}} = \frac{1}{3} \cdot \sqrt{\frac{5}{2}}.$$

EXAMPLE 5 *Two cards are drawn successively with replacement from a well-shuffled pack of* 52 *cards. Find the mean and variance of the number of kings.*

[CBSE 2005C]

SOLUTION Let X be the random variable. Then,

X = number of kings obtained in two draws.

Clearly, X can assume the value 0, 1 or 2.

$$P(\text{drawing a king}) = \frac{4}{52} = \frac{1}{13}.$$

$$P(\text{not drawing a king}) = \left(1 - \frac{1}{13}\right) = \frac{12}{13}.$$

$P(X = 0)$ = P(not a king in the 1st draw and not a king in the 2nd draw)

$$= \left(\frac{12}{13} \times \frac{12}{13}\right) = \frac{144}{169}.$$

$P(X = 1)$ = P(a king in the 1st draw and not a king in the 2nd draw) or P(not a king in the 1st draw and a king in the 2nd draw)

$$= \left(\frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13}\right) = \frac{24}{169}.$$

$P(X = 2)$ = P(a king in the 1st draw and a king in the 2nd draw)

$$= \left(\frac{1}{13} \times \frac{1}{13}\right) = \frac{1}{169}.$$

Hence, the probability distribution is given by

$X = x_i$	0	1	2
p_i	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$\therefore$ mean, $\mu = \Sigma x_i\, p_i = \left(0 \times \frac{144}{169}\right) + \left(1 \times \frac{24}{169}\right) + \left(2 \times \frac{1}{169}\right) = \frac{2}{13}$.

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left[\left(0 \times \frac{144}{169}\right) + \left(1 \times \frac{24}{169}\right) + \left(4 \times \frac{1}{169}\right) - \frac{4}{169}\right] = \frac{24}{169}.$$

EXAMPLE 6 *Two cards are drawn simultaneously (or successively without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of aces.* **[CBSE 2001, '12]**

SOLUTION Let X be the random variable.

Then, X denotes the number of aces in a draw of 2 cards.

$\therefore$ X can assume the value 0, 1 or 2.

Number of ways of drawing 2 cards out of $52 = C(52, 2)$.

$P(X = 0) = P$(both non-aces, i.e., 2 non-aces out of 48)

$$= \frac{{}^{48}C_2}{{}^{52}C_2} = \left(\frac{48 \times 47}{2 \times 1} \times \frac{2}{52 \times 51}\right) = \frac{188}{221}.$$

$P(X = 1) = P$[(one ace out of 4) and (one non-ace out of 48)]

$$= \frac{{}^4C_1 \times {}^{48}C_1}{{}^{52}C_2} = \left(\frac{4 \times 48}{52 \times 51} \times 2\right) = \frac{32}{221}.$$

$$P(X = 2) = P(\text{both aces}) = \frac{{}^4C_2}{{}^{52}C_2} = \left(\frac{4 \times 3}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{1}{221}.$$

Thus, we have the following probability distribution:

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$\therefore$ mean, $\mu = \Sigma x_i\, p_i = \left(0 \times \frac{188}{221}\right) + \left(1 \times \frac{32}{221}\right) + \left(2 \times \frac{1}{221}\right) = \frac{2}{13}$.

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$$= \left(0 \times \frac{188}{221}\right) + \left(1 \times \frac{32}{221}\right) + \left(4 \times \frac{1}{221}\right) - \frac{4}{169}$$

$$= \left(\frac{36}{221} - \frac{4}{169}\right) = \frac{400}{2873}.$$

EXAMPLE 7 *Three defective bulbs are mixed with 7 good ones. Let X be the number of defective bulbs when 3 bulbs are drawn at random. Find the mean and variance of X.*

SOLUTION Let X denote the random variable showing the number of defective bulbs.

Then, X can take the value 0, 1, 2 or 3.

$\therefore \quad P(X=0) = P(\text{none of the bulbs is defective})$

$= P(\text{all the 3 bulbs are good ones})$

$$= \frac{{}^7C_3}{{}^{10}C_3} = \left(\frac{7\times6\times5}{3\times2\times1}\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{7}{24}.$$

$P(X=1) = P(\text{1 defective and 2 non-defective bulbs})$

$$= \frac{{}^3C_1\times{}^7C_2}{{}^{10}C_3} = \left(3\times\frac{7\times6}{2\times1}\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{21}{40}.$$

$P(X=2) = P(\text{2 defective and 1 good one})$

$$= \frac{{}^3C_2\times{}^7C_1}{{}^{10}C_3} = \left(\frac{3\times2}{2\times1}\times7\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{7}{40}.$$

$P(X=3) = P(\text{3 defective bulbs})$

$$= \frac{{}^3C_3}{{}^{10}C_3} = \left(1\times\frac{3\times2\times1}{10\times9\times8}\right) = \frac{1}{120}.$$

Thus, the probability distribution is given by

$X = x_i$	0	1	2	3
p_i	$\frac{7}{24}$	$\frac{21}{40}$	$\frac{7}{40}$	$\frac{1}{120}$

$$\therefore \text{mean}, \mu = \Sigma x_i p_i = \left(0\times\frac{7}{24}\right)+\left(1\times\frac{21}{40}\right)+\left(2\times\frac{7}{40}\right)+\left(3\times\frac{1}{120}\right) = \frac{9}{10}.$$

$$\text{Variance}, \sigma^2 = \Sigma x_i^2 p_i - \mu^2$$

$$= \left(0\times\frac{7}{24}\right)+\left(1\times\frac{21}{40}\right)+\left(4\times\frac{7}{40}\right)+\left(9\times\frac{1}{120}\right)-\frac{81}{100}$$

$$= \left(\frac{13}{10}-\frac{81}{100}\right) = \frac{49}{100}.$$

EXAMPLE 8 *An urn contains 4 white and 3 red balls. Let X be the number of red balls in a random draw of 3 balls. Find the mean and variance of X.*

SOLUTION When 3 balls are drawn at random, there may be no red ball, 1 red ball, 2 red balls or 3 red balls.

Let X denote the random variable showing the number of red balls in a draw of 3 balls.

Then, X can take the value 0, 1, 2 or 3.

$P(X=0)$ = P(getting no red ball)

= P(getting 3 white balls)

$= \frac{{}^4C_3}{{}^7C_3} = \left(\frac{4\times3\times2}{3\times2\times1}\times\frac{3\times2\times1}{7\times6\times5}\right) = \frac{4}{35}$.

$P(X=1)$ = P(getting 1 red and 2 white balls)

$= \frac{{}^3C_1\times{}^4C_2}{{}^7C_3} = \left(\frac{3\times4\times3}{2}\times\frac{3\times2\times1}{7\times6\times5}\right) = \frac{18}{35}$.

$P(X=2)$ = P(getting 2 red and 1 white ball)

$= \frac{{}^3C_2\times{}^4C_1}{{}^7C_3} = \left(\frac{3\times2}{2\times1}\times4\times\frac{3\times2\times1}{7\times6\times5}\right) = \frac{12}{35}$.

$P(X=3) = P(\text{getting 3 red balls}) = \frac{{}^3C_3}{{}^7C_3} = \frac{1\times3\times2\times1}{7\times6\times5} = \frac{1}{35}$.

Thus, the probability distribution of X is given below.

$X = x_i$	0	1	2	3
p_i	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

$\therefore$ mean, $\mu = \Sigma x_i p_i = \left(0\times\frac{4}{35}\right)+\left(1\times\frac{18}{35}\right)+\left(2\times\frac{12}{35}\right)+\left(3\times\frac{1}{35}\right) = \frac{9}{7}$.

Variance, $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$

$= \left[\left(0\times\frac{4}{35}\right)+\left(1\times\frac{18}{35}\right)+\left(4\times\frac{12}{35}\right)+\left(9\times\frac{1}{35}\right)-\frac{81}{49}\right]$

$= \left(\frac{15}{7}-\frac{81}{49}\right) = \frac{24}{49}$.

EXAMPLE 9 *In a game, 3 coins are tossed. A person is paid ₹ 5 if he gets all heads or all tails; and he is supposed to pay ₹ 3 if he gets one head or two heads. What can he expect to win on an average per game?*

SOLUTION In tossing 3 coins, the sample space is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$\therefore$ $n(S) = 8$.

P(getting all heads or all tails) $= \frac{2}{8} = \frac{1}{4}$.

P(getting one head or 2 heads) $= \frac{6}{8} = \frac{3}{4}$.

Let X = number of rupees the person gets.

Then, possible values of X are 5 and –3.

$P(X=5) = \frac{1}{4}$ and $P(X=-3) = \frac{3}{4}$.

Thus, we have

$X = x_i$	5	–3
p_i	$\frac{1}{4}$	$\frac{3}{4}$

$\therefore$ the required expectations = mean, $\mu = \Sigma x_i\ p_i$

$$= \left(5 \times \frac{1}{4}\right) + (-3) \times \frac{3}{4} = -1,$$

i.e., he loses ₹ 1 per toss.

EXERCISE 31

1. Find the mean (μ), variance (σ^2) and standard deviation (σ) for each of the following probability distributions:

(i)

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

[CBSE 2007]

(ii)

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

(iii)

x_i	–3	–1	0	2
p_i	0.2	0.4	0.3	0.1

(iv)

x_i	–2	–1	0	1	2
p_i	0.1	0.2	0.4	0.2	0.1

2. Find the mean and variance of the number of heads when two coins are tossed simultaneously.
3. Find the mean and variance of the number of tails when three coins are tossed simultaneously.
4. A die is tossed twice. 'Getting an odd number on a toss' is considered a success. Find the probability distribution of number of successes. Also, find the mean and variance of the number of successes.
5. A die is tossed twice. 'Getting a number greater than 4' is considered a success. Find the probability distribution of number of successes. Also, find the mean and variance of the number of successes.
6. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of successes. Also, find the mean and variance of number of successes. **[CBSE 2008]**

7. A coin is tossed 4 times. Let X denote the number of heads. Find the probability distribution of X. Also, find the mean and variance of X. **[CBSE 2005]**

8. Let X denote the number of times 'a total of 9' appears in two throws of a pair of dice. Find the probability distribution of X. Also, find the mean, variance and standard deviation of X.

9. There are 5 cards, numbered 1 to 5, one number on each card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two cards drawn. Find the mean and variance of X.

10. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability distribution of number of kings. Also, compute the variance for the number of kings. **[CBSE 2007]**

11. A box contains 16 bulbs, out of which 4 bulbs are defective. Three bulbs are drawn at random from the box. Let X be the number of defective bulbs drawn. Find the mean and variance of X.

12. 20% of the bulbs produced by a machine are defective. Find the probability distribution of the number of defective bulbs in a sample of 4 bulbs chosen at random. **[CBSE 2004C]**

13. Four bad eggs are mixed with 10 good ones. Three eggs are drawn one by one without replacement. Let X be the number of bad eggs drawn. Find the mean and variance of X.

14. Four rotten oranges are accidentally mixed with 16 good ones. Three oranges are drawn at random from the mixed lot. Let X be the number of rotten oranges drawn. Find the mean and variance of X.

15. Three balls are drawn simultaneously from a bag containing 5 white and 4 red balls. Let X be the number of red balls drawn. Find the mean and variance of X.

16. Two cards are drawn without replacement from a well-shuffled deck of 52 cards. Let X be the number of face cards drawn. Find the mean and variance of X.

17. Two cards are drawn one by one with replacement from a well-shuffled deck of 52 cards. Find the mean and variance of the number of aces.

18. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the number of hearts in the three cards drawn. Find the mean and variance of X.

19. Five defective bulbs are accidently mixed with 20 good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution from this lot. **[CBSE 2007]**

ANSWERS (EXERCISE 31)

1. (i) Mean = 1.2, variance = 0.56, SD =0.74

(ii) Mean = 2, variance = 1, SD = 1

(iii) Mean = -0.8, variance = 2.6, SD = 1.612

(iv) Mean = 0, variance = 1.2, SD = 1.095

2. Mean = 1, variance = 0.5

3. Mean =1.5 variance = 0.75

4.

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Mean = 1, variance = 0.5

5.

$X = x_i$	0	1	2
p_i	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Mean $= \frac{2}{3}$, variance $= \frac{4}{9}$

6.

$X = x_i$	0	1	2	3	4
p_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

Mean $= \frac{2}{3}$, variance $= \frac{5}{9}$

7.

$X = x_i$	0	1	2	3	4
p_i	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Mean = 2, variance = 1

8.

$X = x_i$	0	1	2
p_i	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

Mean $= \frac{2}{9}$, variance $= \frac{16}{81}$, SD $= \frac{4}{9}$

9.

$X = x_i$	3	4	5	6	7	8	9
p_i	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Mean = 6, variance = 3

10.

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$\text{Variance} = \frac{400}{2873}$

11. $\text{Mean} = \frac{3}{4}$, $\text{variance} = \frac{39}{80}$

13. $\text{Mean} = \frac{6}{7}$, $\text{variance} = \frac{30}{49}$

14. $\text{Mean} = \frac{3}{5}$, $\text{variance} = \frac{68}{125}$

15. $\text{Mean} = \frac{4}{3}$, $\text{variance} = \frac{5}{9}$

16. $\text{Mean} = \frac{20}{13}$, $\text{variance} = \frac{1000}{2873}$

17. $\text{Mean} = \frac{2}{13}$, $\text{variance} = \frac{24}{169}$

18. $\text{Mean} = \frac{3}{4}$, $\text{variance} = \frac{9}{16}$

19.

X	0	1	2	3	4
$P(X)$	$\frac{969}{2530}$	$\frac{114}{253}$	$\frac{38}{253}$	$\frac{4}{253}$	$\frac{1}{2530}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 31)

2. $S = \{HH, HT, TH, TT\}$.
Let X be the number of heads. Then, $X = 0, 1$ or 2.

$P(X = 0) = P(\text{getting no head}) = \frac{1}{4}$.

$P(X = 1) = P(\text{getting 1 head}) = \frac{2}{4} = \frac{1}{2}$.

$P(X = 2) = P(\text{getting 2 heads}) = \frac{1}{4}$.

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

3.

$X = x_i$	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

4. In a single toss,

$P(\text{success}) = P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}$, and

$P(\text{non-success}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

Let X be the number of successes. Then, $X = 0, 1$ or 2.

$P(X = 0) = P[(\text{non-success in the 1st toss and non-success in the 2nd})]$

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{4}.$$

$P(X = 1) = P$[(success in the 1st toss and non-success in the 2nd)
or (non-success in 1st toss and success in the 2nd)]

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{2}.$$

$P(X = 2) = P$[(success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{4}.$$

$X = x_i$	0	1	2
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

5. In a single toss, $P(\text{success}) = \frac{2}{6} = \frac{1}{3}$ and $P(\text{non-success}) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$.

$P(X = 0) = P$(non-success in the 1st draw and non-success in the second)

$$= \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{4}{9}.$$

$P(X = 1) = P$(success in the 1st toss and non-success in the 2nd)
or (non-success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3}\right) = \frac{4}{9}.$$

$P(X = 2) = P$[(success in the 1st toss and success in the 2nd)]

$$= \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{1}{9}.$$

$X = x_i$	0	1	2
p_i	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

6. In a single throw, $P(\text{doublet}) = \frac{6}{36} = \frac{1}{6}$, and $P(\text{non-doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$.

Let X be the number of doublets. Then, $X = 0, 1, 2$ or 3.

$P(X = 0) = P$(non-doublet in each case)

$$= P(\overline{D}_1 \ \overline{D}_2 \ \overline{D}_3 \ \overline{D}_4) = \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{625}{1296}.$$

$P(X = 1) = P$(one doublet)

$$= P(D_1 \ \overline{D}_2 \ \overline{D}_3 \ \overline{D}_4) \text{ or } P(\overline{D}_1 \ D_2 \ \overline{D}_3 \ \overline{D}_4)$$

$$\text{or } P(\overline{D}_1 \ \overline{D}_2 \ D_3 \ \overline{D}_4) \text{ or } P(\overline{D}_1 \ \overline{D}_2 \ \overline{D}_3 \ D_4)$$

$$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{125}{1296}\right) = \frac{125}{324}.$$

$P(X = 2) = P(\text{two doublets})$

$$= P(D_1\ D_2\ \overline{D}_3\ \overline{D}_4) \quad \text{or } P(D_1\ \overline{D}_2\ D_3\ \overline{D}_4) \text{ or } P(D_1\ \overline{D}_2\ \overline{D}_3\ D_4)$$

$$\text{or } P(\overline{D}_1\ D_2\ D_3\ \overline{D}_4) \text{ or } P(\overline{D}_1\ D_2\ \overline{D}_3\ D_4) \text{ or } P(\overline{D}_1\ \overline{D}_2\ D_3\ D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$+ \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(6 \times \frac{25}{1296}\right) = \frac{25}{216}.$$

$P(X = 3) = P(\text{three doublets})$

$$= P(D_1\ D_2\ D_3\ \overline{D}_4) \text{ or } P(D_1\ D_2\ \overline{D}_3\ D_4)$$

$$\text{or } P(D_1\ \overline{D}_2\ D_3\ D_4) \text{ or } P(\overline{D}_1\ D_2\ D_3\ D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{5}{1296}\right) = \frac{5}{324}.$$

$P(X = 4) = P(\text{four doublets}) = P(D_1\ D_2\ D_3\ D_4)$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{1296}.$$

Thus, we have

$X = x_i$	0	1	2	3	4
p_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

Now, find μ and σ^2.

7. In a single throw of a coin, $P(H) = \frac{1}{2}$ and $P(\overline{H}) = P(T) = \frac{1}{2}$.

Let X show the number of heads. Then, $X = 0, 1, 2, 3$ or 4.

$$P(X = 0) = P(\text{no head}) = P(\overline{H}_1\ \overline{H}_2\ \overline{H}_3\ \overline{H}_4) = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{16}.$$

$P(X = 1) = P(\text{one head}) = P(H_1\ \overline{H}_2\ \overline{H}_3\ \overline{H}_4)$ or $(\overline{H}_1\ H_2\ \overline{H}_3\ \overline{H}_4)$

$$\text{or } P(\overline{H}_1\ \overline{H}_2\ H_3\ \overline{H}_4) \text{ or } P(\overline{H}_1\ \overline{H}_2\ \overline{H}_3\ H_4)$$

$$= 4\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \left(4 \times \frac{1}{16}\right) = \frac{1}{4}.$$

$P(X = 2) = P(\text{two heads}) = P(H_1\ H_2\ \overline{H}_3\ \overline{H}_4)$ or $P(H_1\ \overline{H}_2\ H_3\ \overline{H}_4)$

$$\text{or } P(H_1\ \overline{H}_2\ \overline{H}_3\ H_4) \text{ or } P(\overline{H}_1\ H_2\ H_3\ \overline{H}_4)$$

$$\text{or } P(\overline{H}_1\ H_2\ \overline{H}_3\ H_4) \text{ or } P(\overline{H}_1\ \overline{H}_2\ H_3\ H_4)$$

$$= \left(6 \times \frac{1}{16}\right) = \frac{3}{8}.$$

$P(X = 3) = P(\text{three heads}) = P(H_1 \ \ H_2 \ \ H_3 \ \ \overline{H}_4)$ or $(H_1 \ \ H_2 \ \ \overline{H}_3 \ \ H_4)$

or $P(H_1 \ \ \overline{H}_2 \ \ H_3 \ \ H_4)$ or $P(\overline{H}_1 \ \ H_2 \ \ H_3 \ \ H_4)$

$$= 4 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \left(4 \times \frac{1}{16}\right) = \frac{1}{4}.$$

$$P(X = 4) = P(\text{four heads}) = P(H_1 \ \ H_2 \ \ H_3 \ \ H_4) = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{16}.$$

Thus, we have

$X = x_i$	0	1	2	3	4
p_i	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

8. Let $E = \{(3, 6), (6, 3), (4, 5), (5, 4)\}$. So, $n(E) = 4$.

$\therefore \quad P(E) = \frac{4}{36} = \frac{1}{9}$, and $P(\overline{E}) = \left(1 - \frac{1}{9}\right) = \frac{8}{9}.$

Let X be the number of times 'a total of 9' appears in 2 throws.

Then, $X = 0, 1$ or 2.

$$P(X = 0) = P(\overline{E}_1 \ \overline{E}_2) = \left(\frac{8}{9} \times \frac{8}{9}\right) = \frac{64}{81}.$$

$$P(X = 1) = P[(E_1 \ \overline{E}_2) \text{ or } (\overline{E}_1 \ E_2)] = P(E_1 \ \overline{E}_2) + P(\overline{E}_1 \ E_2)$$

$$= \left(\frac{1}{9} \times \frac{8}{9}\right) + \left(\frac{8}{9} \times \frac{1}{9}\right) = \frac{16}{81}.$$

$$P(X = 2) = P(E_1 \ E_2) = P(E_1) \times P(E_2) = \left(\frac{1}{9} \times \frac{1}{9}\right) = \frac{1}{81}.$$

$X = x_i$	0	1	2
p_i	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

9. $S = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)\}$

Then $X = 3, 4, 5, 6, 7, 8$ and 9.

$$P(X = 3) = \frac{2}{20} = \frac{1}{10}; P(X = 4) = \frac{2}{20} = \frac{1}{10}; P(X = 5) = \frac{4}{20} = \frac{1}{5};$$

$$P(X = 6) = \frac{4}{20} = \frac{1}{5}; P(X = 7) = \frac{4}{20} = \frac{1}{5}; P(X = 8) = \frac{2}{20} = \frac{1}{10};$$

$$P(X = 9) = \frac{2}{20} = \frac{1}{10}.$$

Thus, we have

$X = x_i$	3	4	5	6	7	8	9
p_i	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

10. Let X be the number of kings. Then, $X = 0$, 1 or 2.

$P(X = 0) = P$(none is a king) = P(both are non-kings)

$$= \frac{{}^{48}C_2}{{}^{52}C_2} = \left(\frac{48 \times 47}{2} \times \frac{2}{52 \times 51}\right) = \frac{188}{221}.$$

$P(X = 1) = P$(one king and one non-king)

$$= \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \left(4 \times 48 \times \frac{2}{52 \times 51}\right) = \frac{32}{221}.$$

$P(X = 2) = P$(both are kings)

$$= \frac{{}^{4}C_2}{{}^{52}C_2} = \left(\frac{4 \times 3}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{1}{221}.$$

Thus, we have

$X = x_i$	0	1	2
p_i	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

11. Three bulbs drawn one by one without replacement is the same as drawing 3 bulbs simultaneously.

Let X = number of defective bulbs in a lot of 3 bulbs drawn.

Then, $X = 0, 1, 2$ or 3.

$P(X = 0) = P$(none of the bulbs is defective)

$$= \frac{{}^{12}C_3}{{}^{16}C_3} = \left(\frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{11}{28}.$$

$P(X = 1) = P$(1 defective bulb and 2 nondefective bulbs)

$$= \frac{{}^{4}C_1 \times {}^{12}C_2}{{}^{16}C_3} = \left(\frac{4 \times 12 \times 11}{2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{33}{70}.$$

$P(X = 2) = P$(2 defective bulbs and 1 nondefective bulb)

$$= \frac{({}^{4}C_2 \times {}^{12}C_1)}{{}^{16}C_3} = \left(\frac{4 \times 3}{2 \times 1} \times 12 \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{9}{70}.$$

$P(X = 3) = P$(3 defective bulbs)

$$= \frac{{}^{4}C_3}{{}^{16}C_3} = \left(\frac{4 \times 3 \times 2}{3 \times 2 \times 1} \times \frac{3 \times 2 \times 1}{16 \times 15 \times 14}\right) = \frac{1}{140}.$$

$X = x_i$	0	1	2	3
p_i	$\frac{11}{28}$	$\frac{33}{70}$	$\frac{9}{70}$	$\frac{1}{140}$

12. Let there be 100 bulbs in all and let X be the number of defective bulbs. Then,

$P(X=0) = P(\text{none is defective}) = \frac{{}^{80}C_4}{{}^{100}C_4}$.

$P(X=1) = P(1 \text{ defective and } 3 \text{ nondefective})$

$= \frac{({}^{20}C_1 \times {}^{80}C_3)}{{}^{100}C_4}$.

$P(X=2) = \frac{({}^{20}C_2 \times {}^{80}C_2)}{{}^{100}C_4}$; $P(X=3) = \frac{({}^{20}C_3 \times {}^{80}C_1)}{{}^{100}C_4}$; $P(X=4) = \frac{{}^{20}C_4}{{}^{100}C_4}$.

15. $P(X=0) = \frac{{}^5C_3}{{}^9C_3} = \frac{5}{42}$, $P(X=1) = \frac{({}^4C_1 \times {}^5C_2)}{{}^9C_3} = \frac{10}{21}$,

$P(X=2) = \frac{({}^4C_2 \times {}^5C_1)}{{}^9C_3} = \frac{5}{14}$, $P(X=3) = \frac{{}^4C_3}{{}^9C_3} = \frac{1}{21}$.

16. There are 12 face cards (4 kings, 4 queens and 4 jacks).

Clearly, $X = 0$ or 1 or 2.

$P(X=0) = P(\text{no face card})$

$= P(\text{drawing 2 cards out of 40 non-face cards})$

$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40 \times 39}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{10}{17}$.

$P(X=1) = P(1 \text{ face card and } 1 \text{ non-face card})$

$= \frac{({}^{12}C_1 \times {}^{40}C_1)}{{}^{52}C_2} = \left(12 \times 40 \times \frac{2 \times 1}{52 \times 51}\right) = \frac{80}{221}$.

$P(X=2) = P(2 \text{ face cards})$

$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40 \times 39}{2 \times 1} \times \frac{2 \times 1}{52 \times 51}\right) = \frac{10}{17}$.

Thus, we have

$X = x_i$	0	1	2
p_i	$\frac{10}{17}$	$\frac{80}{221}$	$\frac{10}{17}$

Now, find the mean and variance.

17. See Example 5.

18. There are 13 hearts and 39 other cards.

Let E = event of drawing a heart.

Then, $P(E) = \frac{13}{52} = \frac{1}{4}$ and $P(\bar{E}) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$.

Let X = number of hearts in a draw.

Then, $X = 0, 1, 2$ or 3.

$$P(X=0) = P(\overline{E}\ \overline{E}\ \overline{E}) = P(\overline{E}) \times P(\overline{E}) \times P(\overline{E}) = \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{27}{64}.$$

$$P(X=1) = P[(E\ \overline{E}\ \overline{E}) \text{ or } (\overline{E}\ E\ \overline{E}) \text{ or } (\overline{E}\ \overline{E}\ E)]$$

$$= P(E\ \overline{E}\ \overline{E}) + P(\overline{E}\ E\ \overline{E}) + P(\overline{E}\ \overline{E}\ E)$$

$$= \left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}\right) = \frac{27}{64}.$$

$$P(X=2) = P[(E\ E\ \overline{E}) \text{ or } (E\ \overline{E}\ E) \text{ or } (\overline{E}\ E\ E)]$$

$$= P(E\ E\ \overline{E}) + P(E\ \overline{E}\ E) + P(\overline{E}\ E\ E)$$

$$= \left(\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{9}{64}.$$

$$P(X=3) = P(E\ E\ E) = P(E) \times P(E) \times P(E) = \left(\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{1}{64}.$$

$X = x_i$	0	1	2	3
p_i	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

Find the mean and variance.

32. BINOMIAL DISTRIBUTION

SUCCESS AND FAILURE IN AN EXPERIMENT There are certain kinds of experiments which have two possible outcomes. One of these two outcomes is called a *success,* while the other is called a *failure.*

For example, in tossing a coin, we get either a head or a tail. If getting head is taken as a success then getting a tail is a failure.

BERNOULLI'S THEOREM *Let there be n independent trials in an experiment and let the random variable X denote the number of successes in these trials. Let the probability of getting a success in a single trial be p and that of getting a failure be q so that* $p + q = 1$. *Then,*

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}.$$

PROOF Let us denote a success by S and a failure by F.
Number of ways of getting r successes in n trials $= {}^nC_r$.

$$\therefore \quad P(X = r) = {}^nC_r \cdot \left\{ \underbrace{SSS \ldots S}_{r \text{ times}} \text{ and } \underbrace{FFF \ldots F}_{(n-r) \text{ times}} \right\}$$

$$= {}^nC_r \cdot \{P(S) \cdot P(S) \ldots r \text{ times}\} \times \{P(F) \cdot P(F) \ldots (n-r) \text{ times}\}$$

$$= {}^nC_r \cdot (p \cdot p \cdot p \ldots r \text{ times}) \times [q \cdot q \cdot q \ldots (n-r) \text{ times}]$$

$$= {}^nC_r \cdot p^r \cdot q^{(n-r)}.$$

Hence, $P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}$.

REMARK We have

$$P(X = 0) = q^n;\ P(X = 1) = npq^{(n-1)};\ P(X = 2) = {}^nC_2 \cdot p^2 \cdot q^{(n-2)}, \text{ etc.}$$

The probability distribution of X may be expressed as

$$\begin{pmatrix} X: & 0 & 1 & \ldots & r \\ P(X): & q^n & npq^{(n-1)} & \ldots & C_r \cdot p^r \cdot q^{(n-r)} \end{pmatrix}.$$

This distribution is called a *binomial distribution.*

Conditions for the Applicability of a Binomial Distribution

(i) The experiment is performed for a finite and fixed number of trials.

(ii) Each trial must give either a success or a failure.

(iii) The probability of a success in each trial is the same.

SOLVED EXAMPLES

EXAMPLE 1 *A coin is tossed 4 times. If X is the number of heads observed, find the probability distribution of X.*

SOLUTION When a coin is tossed, we have $S = \{H, T\}$.

$$P(\text{getting a head}) = \frac{1}{2}, \text{ and}$$

$$P(\text{not getting a head}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

Let X be the random variable denoting the number of heads.

In 4 trials, we may get 0 or 1 or 2 or 3 or 4 heads.

So, X may assume the values 0, 1, 2, 3, 4.

$$P(X = 0) = {}^4C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{(4-0)} = \frac{1}{16}.$$

$$P(X = 1) = {}^4C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^{(4-1)} = \frac{1}{4}.$$

$$P(X = 2) = {}^4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{(4-2)} = \frac{3}{8}.$$

$$P(X = 3) = {}^4C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{(4-3)} = \frac{1}{4}.$$

$$P(X = 4) = {}^4C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{(4-4)} = \frac{1}{16}.$$

Hence, the required probability distribution is given by

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 & 4 \\ P(X): & \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \end{pmatrix}.$$

EXAMPLE 2 *Find the probability distribution of the number of sixes in three tosses of a die.*

SOLUTION When a die is tossed, we have $S = \{1, 2, 3, 4, 5, 6\}$.

$$\therefore \quad P(\text{getting a six}) = \frac{1}{6} \text{ and } P(\text{not getting a six}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Let X be the random variable denoting the number of sixes.

In 3 trials, the number of sixes may be 0 or 1 or 2 or 3.

So, X may assume the values 0, 1, 2, 3.

$$P(X = 0) = {}^3C_0 \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{(3-0)} = \frac{125}{216}.$$

$$P(X = 1) = {}^3C_1 \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^{(3-1)} = \frac{25}{72}.$$

$$P(X=2) = {}^3C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{(3-2)} = \frac{5}{72}.$$

$$P(X=3) = {}^3C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^{(3-3)} = \frac{1}{216}.$$

The required probability distribution of X is given below:

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 \\ P(X): & \frac{125}{216} & \frac{25}{72} & \frac{5}{72} & \frac{1}{216} \end{pmatrix}.$$

EXAMPLE 3 *Find the probability distribution of the number of doublets in four throws of a pair of dice.*

SOLUTION When a pair of dice is thrown, there are 36 possible outcomes.

$\therefore\ n(S) = 36.$

All possible doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$$P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}, \text{ and}$$

$$P(\text{not getting a doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Let X denote the number of doublets.

In 4 throws, we can have 0 or 1 or 2 or 3 or 4 doublets.

$$P(X=0) = {}^4C_0 \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^4 = \frac{625}{1296}.$$

$$P(X=1) = {}^4C_1 \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^3 = \frac{125}{324}.$$

$$P(X=2) = {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 = \frac{25}{216}.$$

$$P(X=3) = {}^4C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^1 = \frac{5}{324}.$$

$$P(X=4) = {}^4C_4 \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^0 = \frac{1}{1296}.$$

The required probability distribution is given below:

$$\begin{pmatrix} X: & 0 & 1 & 2 & 3 & 4 \\ P(X): & \frac{625}{1296} & \frac{125}{324} & \frac{25}{216} & \frac{5}{324} & \frac{1}{1296} \end{pmatrix}.$$

EXAMPLE 4 *An unbiased coin is tossed 6 times. Find, using binomial distribution, the probability of getting at least 5 heads.* **[CBSE 2000]**

SOLUTION In a single throw of a coin, we have $S = \{H, T\}$.

$$P(\text{getting a head}) = \frac{1}{2}, \text{ and } P(\text{not getting a head}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$\therefore \quad p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$$P(X = r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{6}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(6-r)} = {}^{6}C_r \cdot \left(\frac{1}{2}\right)^6$$

$\therefore \quad P$(getting at least 5 heads)

$$= P(X \geq 5)$$
$$= P(X = 5) + P(X = 6)$$
$$= {}^{6}C_5 \cdot \left(\frac{1}{2}\right)^6 + {}^{6}C_6 \cdot \left(\frac{1}{2}\right)^6 = \left(\frac{3}{32} + \frac{1}{64}\right) = \frac{7}{64}.$$

Hence, the required probability is $\frac{7}{64}$.

EXAMPLE 5 *An unbiased coin is tossed 8 times. Find, by using binomial distribution, the probability of getting at least 3 heads.* **[CBSE 2000]**

SOLUTION In a single throw of a coin, we have $S = \{H, T\}$.

P(getting a head) $= \frac{1}{2}$, and P(not getting a head) $= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$$P(X = r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{8}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(8-r)} = {}^{8}C_r \left(\frac{1}{2}\right)^8$$

$\therefore \quad P$(getting at least 3 heads)

$$= P(X \geq 3)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[{}^{8}C_0 \cdot \left(\frac{1}{2}\right)^8 + {}^{8}C_1 \cdot \left(\frac{1}{2}\right)^8 + {}^{8}C_2 \cdot \left(\frac{1}{2}\right)^8 \right]$$
$$= 1 - \frac{1}{256} \cdot (1 + 8 + 28) = \left(1 - \frac{37}{256}\right) = \frac{219}{256}.$$

Hence, the required probability is $\frac{219}{256}$.

EXAMPLE 6 *Six coins are tossed simultaneously. Find the probability of getting*
(i) 3 heads (ii) no head (iii) at least one head
(iv) not more than 3 heads. **[CBSE 2003]**

SOLUTION The experiment may be taken as throwing a single coin 6 times.
In a single throw of a coin, we have $S = \{H, T\}$.

P(getting a head) $= \frac{1}{2}$, and P(not getting a head) $= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

Let X be the random variable showing the number of heads.

$$P(X=r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{6}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{6-r} = {}^{6}C_r \cdot \left(\frac{1}{2}\right)^6$$

(i) $P(\text{getting 3 heads}) = P(X=3) = {}^{6}C_3 \cdot \left(\frac{1}{2}\right)^6 = \frac{5}{16}$.

(ii) $P(\text{getting no head}) = P(X=0) = {}^{6}C_0 \cdot \left(\frac{1}{2}\right)^6 = \frac{1}{64}$.

(iii) $P(\text{getting at least 1 head})$

$$= 1 - P(X=0) = 1 - \left[{}^{6}C_0 \cdot \left(\frac{1}{2}\right)^6\right]$$

$$= \left(1 - \frac{1}{64}\right) = \frac{63}{64}.$$

(iv) $P(\text{getting not more than 3 heads})$

$$= P(\text{no head or 1 head or 2 heads or 3 heads})$$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= {}^{6}C_0 \cdot \left(\frac{1}{2}\right)^6 + {}^{6}C_1 \cdot \left(\frac{1}{2}\right)^6 + {}^{6}C_2 \cdot \left(\frac{1}{2}\right)^6 + {}^{6}C_3 \cdot \left(\frac{1}{2}\right)^6$$

$$= \left(\frac{1}{2}\right)^6 \cdot (1+6+15+20) = \left(\frac{1}{64} \times 42\right) = \frac{21}{32}.$$

EXAMPLE 7 *A die is thrown 5 times. If getting an odd number is a success, find the probability of getting at least 4 successes.*

SOLUTION When a die is thrown, we have $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore$ $P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}$.

$\therefore$ $P(\text{a success}) = \frac{1}{2}$, and $P(\text{not a success}) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

Let X be the random variable showing the number of successes.

$$P(X=r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{5}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(5-r)} = {}^{5}C_r \cdot \left(\frac{1}{2}\right)^5.$$

$P(\text{at least 4 successes}) = P(4 \text{ successes or } 5 \text{ successes})$

$$= P(X=4) + P(X=5)$$

$$= {}^{5}C_4 \cdot \left(\frac{1}{2}\right)^5 + {}^{5}C_5 \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{6}{32} = \frac{3}{16}.$$

EXAMPLE 8 *In 4 throws with a pair of dice, what is the probability of throwing doublets at least twice?*

SOLUTION In a single throw of a pair of dice, the number of all possible outcomes is 36.

All doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}$, and

$P(\text{not getting a doublet}) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$.

Let X be the random variable denoting the number of doublets.

Then, $P(X = r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{4}C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(4-r)}$.

$P(\text{at least 2 doublets})$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= {}^{4}C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{(4-2)} + {}^{4}C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^{(4-3)} + {}^{4}C_4 \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^{(4-4)}$

$= 6 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 + 4 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^1 + \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^0$

$= \left(\frac{25}{216} + \frac{5}{324} + \frac{1}{1296}\right) = \frac{171}{1296}$.

EXAMPLE 9 *The bulbs produced in a factory are supposed to contain* 5% *defective bulbs. What is the probability that a sample of* 10 *bulbs will contain not more than* 2 *defective bulbs?*

SOLUTION $P(\text{getting a defective bulb}) = \frac{5}{100} = \frac{1}{20}$, and

$P(\text{getting a nondefective bulb}) = \left(1 - \frac{1}{20}\right) = \frac{19}{20}$.

Then, $p = \frac{1}{20}$ and $q = \frac{19}{20}$.

Let X denote the number of defective bulbs.

$P(X = r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{10}C_r \cdot \left(\frac{1}{20}\right)^r \cdot \left(\frac{19}{20}\right)^{(10-r)}$

$P(\text{getting not more than 2 defective bulbs})$

$= P(X = 0 \text{ or } X = 1 \text{ or } X = 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= {}^{10}C_0 \cdot \left(\frac{1}{20}\right)^0 \cdot \left(\frac{19}{20}\right)^{10} + {}^{10}C_1 \cdot \left(\frac{1}{20}\right)^1 \cdot \left(\frac{19}{20}\right)^9 + {}^{10}C_2 \cdot \left(\frac{1}{20}\right)^2 \cdot \left(\frac{19}{20}\right)^8$

$= \left(\frac{19}{20}\right)^{10} + \frac{1}{2} \cdot \left(\frac{19}{20}\right)^9 + \frac{9}{80} \cdot \left(\frac{19}{20}\right)^8 = \left(\frac{19}{20}\right)^8 \cdot \left(\frac{149}{100}\right)$.

Let $A = \left(\frac{19}{20}\right)^8 \cdot \left(\frac{149}{100}\right)$. Then,

$\log A = 8(\log 19 - \log 20) + \log 149 - \log 100$

$$= 8(1.2788 - 1.3010) + 2.1732 - 2$$
$$= -0.0044 = \bar{1}.9956.$$

$\therefore \quad A = \text{antilog}\,(\bar{1}.9956) = 0.99.$

Hence, the required probability $= \frac{99}{100}$.

EXAMPLE 10 *If on an average, out of* 10 *ships, one gets drowned then what is the probability that out of* 5 *ships at least* 4 *reach the shore safely?*

SOLUTION Probability of a ship to reach the shore safely $= \frac{9}{10}$.

Probability that a ship gets drowned $= \left(1 - \frac{9}{10}\right) = \frac{1}{10}$.

Let X be the random variable showing the number of ships reaching the shore safely.

$\therefore \quad P$(at least 4 reaching safely)

$= P$(4 reaching safely or 5 reaching safely)

$= P$(4 reaching safely) + P(5 reaching safely)

$= P(X = 4) + P(X = 5)$

$$= {}^5C_4 \cdot \left(\frac{9}{10}\right)^4 \cdot \left(\frac{1}{10}\right)^{(5-4)} + {}^5C_5 \cdot \left(\frac{9}{10}\right)^5 \cdot \left(\frac{1}{10}\right)^0$$

$$= \frac{1}{2} \cdot \left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^4 \left(\frac{1}{2} + \frac{9}{10}\right) = \frac{7}{5} \cdot \left(\frac{9}{10}\right)^4.$$

Let $A = \frac{7}{5} \cdot \left(\frac{9}{10}\right)^4 = \frac{7 \times (9)^4}{5 \times (10)^4}$

$\Rightarrow \quad \log A = \log 7 + 4 \times \log 9 - \log 5 - 4 \times \log 10$

$= (0.8451 + 4 \times 0.9542 - 0.6990 - 4)$

$= -0.0371 = \bar{1}.9629$

$\Rightarrow \quad A = \text{antilog}\ (\bar{1}.9629) = 0.9181.$

Hence, the required probability is 0.9181.

Mean and Variance of a Binomial Distribution

MEAN *If a random variable* X *assumes the values* $x_1, x_2, \ldots, x_n$ *with probabilities* $p_1, p_2, \ldots, p_n$ *respectively then the mean of* X *is defined by*

$$\mu = \sum_{i=1}^{n} x_i\, p_i.$$

To Find the Mean of a Binomial Distribution

For the binomial distribution

$$P(X = r) = P(r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}, \text{ where } r = 0, 1, 2, \ldots, n$$

the *mean*, μ, is given by

$$\mu = \sum_{r=0}^{n} r \cdot p(r) = \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)}$$

$$= {}^nC_1 \cdot p \cdot q^{n-1} + 2 \cdot {}^nC_2 \cdot p^2 \cdot q^{(n-2)} + \ldots + n \cdot {}^nC_n \cdot p^n q^0$$

$$= 1 \cdot np \cdot q^{n-1} + n(n-1) \cdot p^2 \cdot q^{(n-2)} + \ldots + np^n$$

$$= np \cdot [{}^{(n-1)}C_0 \cdot p^0 \cdot q^{(n-1)} + {}^{(n-1)}C_1 \cdot p^1 \cdot q^{(n-2)} + \ldots + {}^{(n-1)}C_{(n-1)} \cdot p^{n-1} \cdot q^0]$$

$$= (np) \cdot (q + p)^{n-1} = (np) \qquad [\because \quad q + p = 1].$$

Hence, the *mean* is given by $\mu = np$.

The *variance* $= \sigma^2$ is given by

$$\sigma^2 = \sum_{r=0}^{n} r^2 \cdot p(r) - (\text{mean})^2$$

$$= \sum_{r=0}^{n} r^2 \cdot {}^nC_r \cdot p^r q^{(n-r)} - (np)^2 \qquad [\because \quad \text{mean} = np]$$

$$= \sum_{r=0}^{n} \{r + r(r-1)\} \cdot {}^nC_r \, p^r q^{(n-r)} - (np)^2$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r \, p^r q^{(n-r)} + \sum_{r=0}^{n} r(r-1) \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)} - (np)^2$$

$$= np + \sum_{r=2}^{n} r(r-1) \cdot \frac{n(n-1)}{r(r-1)} \cdot {}^{(n-2)}C_{(r-2)} \cdot p^2 \cdot p^{(r-2)} \cdot q^{(n-r)} - (np)^2$$

$$\left[\because \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{(n-r)} = \text{mean} = np \right]$$

$$= np + n(n-1) \cdot p^2 \cdot \left(\sum_{r=0}^{n} {}^{(n-2)}C_{(r-2)} \cdot p^{(r-2)} \cdot q^{(n-r)} \right) - n^2 p^2$$

$$= np + n(n-1) \, p^2 (q + p)^{(n-2)} - n^2 p^2$$

$$= np + n(n-1) \, p^2 - n^2 p^2 \qquad [\because \quad q + p = 1]$$

$$= np - np^2 = np(1 - p) = npq.$$

Hence, *variance* $= npq$.

$\therefore$ standard deviation $= \sqrt{npq}$.

Recurrence Relation for a Binomial Distribution

We have

$$P(r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} \text{ and } P(r+1) = {}^nC_{(r+1)} \cdot p^{(r-1)} \cdot q^{n-r-1}.$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{(r-1)} \cdot p^{(r-1)} \cdot q^{n-r-1}}{{}^nC_r \cdot p^r \cdot q^{(n-r)}}$$

$$= \frac{(n!)}{(r+1)! \cdot (n-r-1)!} \cdot \frac{(r)! \cdot (n-r)!}{(n)!} \cdot \frac{p}{q} = \frac{(n-r)}{(r+1)} \cdot \frac{p}{q}$$

$$\therefore \quad P(r+1) = \frac{(n-r)}{(r+1)} \cdot \frac{p}{q} \cdot P(r) \cdot$$

EXAMPLE 11 *If X follows a binomial distribution with mean 3 and variance $(3/2)$, find (i) $P(X \geq 1)$ (ii) $P(X \leq 5)$.* **[CBSE 2001C]**

SOLUTION We know that mean $= np$ and variance $= npq$.

$$\therefore \quad np = 3 \text{ and } npq = \frac{3}{2} \Rightarrow 3q = \frac{3}{2} \Rightarrow q = \frac{1}{2}.$$

$$\therefore \quad p = (1-q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$$\text{Now, } np = 3 \text{ and } p = \frac{1}{2} \Rightarrow n \times \frac{1}{2} = 3 \Rightarrow n = 6.$$

So, the binomial distribution is given by

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(6-r)} = {}^6C_r \left(\frac{1}{2}\right)^6.$$

(i) $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}.$$

(ii) $P(X \leq 5) = 1 - P(X = 6)$

$$= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}.$$

EXAMPLE 12 *If X follows a binomial distribution with mean 4 and variance 2, find $P(X \geq 5)$.* **[CBSE 2001C]**

SOLUTION We know that mean $= np$ and variance $= npq$.

$$\therefore \quad np = 4 \text{ and } npq = 2.$$

$$\text{Now, } np = 4 \text{ and } npq = 2 \Rightarrow 4q = 2 \Rightarrow q = \frac{1}{2}.$$

$$\therefore \quad p = (1-q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$$\text{Now, } np = 4 \text{ and } p = \frac{1}{2} \Rightarrow \frac{1}{2}n = 4 \Rightarrow n = 8.$$

So, the binomial distribution is given by

$$P(X=r) = {}^{n}C_r \cdot p^r \cdot q^{(n-r)} = {}^{8}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(8-r)} = {}^{8}C_r \cdot \left(\frac{1}{2}\right)^8.$$

$$\therefore \quad P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$$

$$= {}^{8}C_5 \cdot \left(\frac{1}{2}\right)^8 + {}^{8}C_6 \cdot \left(\frac{1}{2}\right)^8 + {}^{8}C_7 \cdot \left(\frac{1}{2}\right)^8 + {}^{8}C_8 \cdot \left(\frac{1}{2}\right)^8$$

$$= [{}^{8}C_3 + {}^{8}C_2 + {}^{8}C_1 + 1]\left(\frac{1}{2}\right)^8$$

$$= (56 + 28 + 8 + 1) \cdot \frac{1}{256} = \frac{93}{256}.$$

EXAMPLE 13 *Find the binomial distribution for which the mean and variance are* 12 *and* 3 *respectively.* **[CBSE 2004C]**

SOLUTION Let X be a binomial variate for which mean = 12 and variance = 3.

Then, $np = 12$ and $npq = 3 \Leftrightarrow 12 \times q = 3 \Leftrightarrow q = \frac{1}{4}$.

$$\therefore \quad p = (1-q) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

And, $np = 12 \Leftrightarrow n = \frac{12}{p} = \left(12 \times \frac{4}{3}\right) = 16.$

Thus, $n = 16$, $p = \frac{3}{4}$ and $q = \frac{1}{4}$.

Hence, the binomial distribution is given by

$$P(X=r) = {}^{16}C_r \cdot \left(\frac{3}{4}\right)^r \cdot \left(\frac{1}{4}\right)^{(16-r)}, \text{ where } r = 0, 1, 2, 3, \ldots, 15.$$

EXAMPLE 14 *If the sum of the mean and variance of a binomial distribution for* 5 *trials is* 1.8, *find the distribution.* **[CBSE 2004]**

SOLUTION We know that

$$\text{mean} = np \text{ and variance} = npq.$$

It is being given that $n = 5$ and mean + variance = 1.8.

$\therefore \quad np + npq = 1.8$, where $n = 5$

$\Leftrightarrow \quad 5p + 5pq = 1.8$

$\Leftrightarrow \quad p + p(1-p) = 0.36 \quad [\because \quad q = (1-p)]$

$\Leftrightarrow \quad p^2 - 2p + 0.36 = 0$

$\Leftrightarrow \quad 100p^2 - 200p + 36 = 0$

$\Leftrightarrow \quad 25p^2 - 50p + 9 = 0$

$\Leftrightarrow \quad 25p^2 - 45p - 5p + 9 = 0$

$\Leftrightarrow \quad 5p(5p-9)-(5p-9)=0$

$\Leftrightarrow \quad (5p-9)(5p-1)=0$

$\Leftrightarrow \quad p=\frac{1}{5}=0.2 \quad [\because \quad p \text{ cannot exceed } 1].$

Thus, $n=5$, $p=0.2$, and $q=(1-p)=(1-0.2)=0.8$.

Let X denote the binomial variate. Then, the required distribution is

$$P(X=r)={}^{n}C_r \cdot p^r \cdot q^{(n-r)}={}^{5}C_r \cdot (0.2)^r \cdot (0.8)^{(5-r)},$$

where $r=0, 1, 2, 3, 4, 5$.

EXAMPLE 15 *The sum and the product of the mean and variance of a binomial distribution are* 24 *and* 128 *respectively. Find the distribution.*

SOLUTION We have

$np+npq=24$ and $np \times npq=128$

$\Leftrightarrow \quad (np)(1+q)=24$ and $n^2p^2 \times q=128$

$\Leftrightarrow \quad n^2p^2=\frac{576}{(1+q)^2}$ and $n^2p^2 \times q=128$

$\Leftrightarrow \quad \frac{576}{(1+q)^2}=\frac{128}{q} \Leftrightarrow 2(1+q^2+2q)=9q$

$\Leftrightarrow \quad 2q^2-5q+2=0 \Leftrightarrow (2q-1)(q-2)=0$

$\Leftrightarrow \quad q=\frac{1}{2} \quad [\because \quad q \neq 2].$

$\therefore \quad p=(1-q)=\left(1-\frac{1}{2}\right)=\frac{1}{2}.$

Now, $np(1+q)=24 \Leftrightarrow n \times \frac{1}{2}\left(1+\frac{1}{2}\right)=24 \Leftrightarrow n=32.$

Hence, the required probability distribution is given by

$$P(X=r)={}^{32}C_r \cdot \left(\frac{1}{2}\right)^{32}.$$

EXAMPLE 16 *In a binomial distribution, prove that*

mean > variance.

SOLUTION Let X be a binomial variate with parameters n and p. Then,

mean $=np$ and variance $=npq$.

$\therefore \quad$ (mean) $-$ (variance) $=(np-npq)=np(1-q)=np^2>0$

$[\because \quad (1-q)=p \text{ and } np^2>0 \text{ as } n \in N]$

$\Rightarrow \quad [(\text{mean})-(\text{variance})]>0$

$\Rightarrow \quad$ mean > variance.

Hence, mean > variance.

EXAMPLE 17 *A die is tossed thrice. Getting an even number is considered a success. What is the variance of the binomial distribution?*

SOLUTION Here, $n = 3$.

Let p = probability of getting an even number in a single throw

$$\Rightarrow \quad p = \frac{3}{6} = \frac{1}{2}.$$

$$\Rightarrow \quad q = (1 - p) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

$$\therefore \quad \text{variance} = npq = \left(3 \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}.$$

EXAMPLE 18 *A die is rolled 20 times. Getting a number greater than 4 is a success. Find the mean and variance of the number of successes.*

SOLUTION In a single throw of a die, we have

$$p = \text{probability of getting a number greater than } 4 = \frac{2}{6} = \frac{1}{3}.$$

$$\therefore \quad q = (1 - p) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$$

Also, $n = 20$ (given).

$$\therefore \quad \text{mean} = np = \left(20 \times \frac{1}{3}\right) = 6.67, \text{ and}$$

$$\text{variance} = npq = \left(20 \times \frac{1}{3} \times \frac{2}{3}\right) = 4.44.$$

EXAMPLE 19 *A die is tossed 180 times. Find the expected number (μ) of times the face with the number 5 will appear. Also, find the standard deviation (σ), and variance (σ^2).*

SOLUTION In a single throw of a die, $S = \{1, 2, 3, 4, 5, 6\}$.

$$p = (\text{probability of getting the number } 5) = \frac{1}{6}.$$

$$\therefore \quad q = (1 - p) = \left(1 - \frac{1}{6}\right) = \frac{5}{6}.$$

Thus, $n = 180$, $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

$$\therefore \quad \mu = np = \left(180 \times \frac{1}{6}\right) = 30.$$

$$\text{Variance} = \sigma^2 = npq = \left(180 \times \frac{1}{6} \times \frac{5}{6}\right) = 25.$$

$$\text{Standard deviation} = \sigma = \sqrt{25} = 5.$$

EXERCISE 32

1. A coin is tossed 6 times. Find the probability of getting at least 3 heads.
2. A coin is tossed 5 times. What is the probability that a head appears an even number of times?
3. 7 coins are tossed simultaneously. What is the probability that a tail appears an odd number of times?
4. A coin is tossed 6 times. Find the probability of getting
   (i) exactly 4 heads (ii) at least 1 head (iii) at most 4 heads.
5. 10 coins are tossed simultaneously. Find the probability of getting
   (i) exactly 3 heads (ii) not more than 4 heads
   (iii) at least 4 heads.
6. A die is thrown 6 times. If 'getting an even number' is a success, find the probability of getting
   (i) exactly 5 successes (ii) at least 5 successes
   (iii) at most 5 successes.
7. A die is thrown 4 times. 'Getting a 1 or a 6' is considered a success. Find the probability of getting
   (i) exactly 3 successes (ii) at least 2 successes
   (iii) at most 2 successes.
8. Find the probability of a 4 turning up at least once in two tosses of a fair die.
9. A pair of dice is thrown 4 times. If 'getting a doublet' is considered a success, find the probability of getting 2 successes.
10. A pair of dice is thrown 7 times. If 'getting a total of 7' is considered a success, find the probability of getting
    (i) no success (ii) exactly 6 successes
    (iii) at least 6 successes (iv) at most 6 successes.
11. There are 6% defective items in a large bulk of items. Find the probability that a sample of 8 items will include not more than one defective item.
12. In a box containing 60 bulbs, 6 are defective. What is the probability that out of a sample of 5 bulbs (i) none is defective, (ii) exactly 2 are defective?
13. The probability that a bulb produced by a factory will fuse after 6 months of use is 0.05. Find the probability that out of 5 such bulbs
    (i) none will fuse after 6 months of use
    (ii) at least one will fuse after 6 months of use
    (iii) not more than one will fuse after 6 months of use.
14. In the items produced by a factory, there are 10% defective items. A sample of 6 items is randomly chosen. Find the probability that this sample contains (i) exactly 2 defective items, (ii) not more than 2 defective items, (iii) at least 3 defective items.

15. Assume that on an average one telephone number out of 15, called between 3 p.m. and 4 p.m. on weekdays, will be busy. What is the probability that if six randomly selected telephone numbers are called, at least 3 of them will be busy?

16. Three cars participate in a race. The probability that any one of them has an accident is 0.1. Find the probability that all the cars reach the finishing line without any accident.

17. Past records show that 80% of the operations performed by a certain doctor were successful. If he performs 4 operations in a day, what is the probability that at least 3 operations will be successful?

18. The probability of a man hitting a target is (1/4). If he fires 7 times, what is the probability of his hitting the target at least twice?

19. In a hurdles race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is (5/6). What is the probability that he will knock down fewer than 2 hurdles?

20. A man can hit a bird, once in 3 shots. On this assumption he fires 3 shots. What is the chance that at least one bird is hit?

21. If the probability that a man aged 60 will live to be 70 is 0.65, what is the probability that out of 10 men, now 60, at least 8 will live to be 70?

22. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one with replacement, what is the probability that (i) none is white, (ii) all are white, (iii) at least one is white?

23. A policeman fires 6 bullets at a burglar. The probability that the burglar will be hit by a bullet is 0.6. What is the probability that the burglar is still unhurt?

24. A die is tossed thrice. A success is 1 or 6 on a toss. Find the mean and variance of successes.

25. A die is thrown 100 times. Getting an even number is considered a success. Find the mean and variance of successes.

26. Determine the binomial distribution whose mean is 9 and variance is 6.

[CBSE 2006]

27. Find the binomial distribution whose mean is 5 and variance is 2.5.

28. The mean and variance of a binomial distribution are 4 and (4/3) respectively. Find $P(X \geq 1)$. **[CBSE 2004]**

29. For a binomial distribution, the mean is 6 and the standard deviation is $\sqrt{2}$. Find the probability of getting 5 successes.

30. In a binomial distribution, the sum and the product of the mean and the variance are (25/3) and (50/3) respectively. Find the distribution.

31. Obtain the binomial distribution whose mean is 10 and standard deviation is $2\sqrt{2}$.

32. Bring out the fallacy, if any, in the following statement:
'The mean of a binomial distribution is 6 and its variance is 9'.

ANSWERS (EXERCISE 32)

1. $\frac{21}{32}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$

4. (i) $\frac{15}{64}$ (ii) $\frac{63}{64}$ (iii) $\frac{57}{64}$ **5.** (i) $\frac{15}{128}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

6. (i) $\frac{3}{32}$ (ii) $\frac{7}{64}$ (iii) $\frac{63}{64}$ **7.** (i) $\frac{8}{81}$ (ii) $\frac{11}{27}$ (iii) $\frac{8}{9}$

8. $\frac{11}{36}$ **9.** $\frac{25}{216}$ **10.** (i) $\left(\frac{5}{6}\right)^7$ (ii) $\frac{35}{6^7}$ (iii) $\frac{1}{6^5}$ (iv) $\left(1-\frac{1}{6^7}\right)$

11. $\left(\frac{47}{50}\right)^7 \times \left(\frac{71}{50}\right)$ **12.** (i) $\left(\frac{9}{10}\right)^5$ (ii) $\frac{729}{10000}$

13. (i) $\left(\frac{19}{20}\right)^5$ (ii) $1-\left(\frac{19}{20}\right)^5$ (iii) $\left(\frac{19}{20}\right)^4 \times \left(\frac{6}{5}\right)$

14. (i) $\frac{3}{20} \times \left(\frac{9}{10}\right)^4$ (ii) $\frac{3}{2} \times \left(\frac{19}{20}\right)^4$ (iii) $1-\left[\frac{3}{2} \times \left(\frac{19}{20}\right)^4\right]$

15. $1-\left(\frac{14}{15}\right)^4 \cdot \left(\frac{59}{45}\right)$ **16.** $\frac{729}{1000}$ **17.** $\frac{512}{625}$

18. $\frac{4547}{8192}$ **19.** $\frac{5^{10}}{(2 \times 6^9)}$ **20.** $\frac{19}{27}$

21. 0.2615 **22.** (i) $\frac{81}{256}$ (ii) $\frac{1}{256}$ (iii) $\frac{175}{256}$

23. 0.004096 **24.** $\mu = 1,\ \sigma^2 = \frac{2}{3}$ **25.** $\mu = 50,\ \sigma^2 = 25$

26. ${}^{27}C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(27-r)}$, where $r = 0, 1, 2, 3, \ldots, 27$

27. ${}^{10}C_r \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{(10-r)}$, $0 \le r \le 10$ **28.** $\frac{728}{729}$

29. ${}^{9}C_5 \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^4$ **30.** ${}^{15}C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(15-r)}$

31. ${}^{50}C_r \cdot \left(\frac{1}{5}\right)^r \cdot \left(\frac{4}{5}\right)^{(50-r)}$, $0 \le r \le 50$

32. The probability of getting a failure (i.e., q) cannot be greater than 1.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 32)

1. In a single toss, $P(H)=\frac{1}{2}$ and $P(\text{not } H)=\frac{1}{2}$.

   $\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=6$.

   Let X show the number of heads. Then,

   $$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{6}C_r\left(\frac{1}{2}\right)^6.$$

   Required probability $=P(X=3)+P(X=4)+P(X=5)+P(X=6)$.

2. Here, $p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=5$.

   Let X show the number of heads. Then,

   $$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{5}C_r\left(\frac{1}{2}\right)^5.$$

   Required probability $=P(X=0)+P(X=2)+P(X=4)$.

3. 7 coins being tossed simultaneously is the same as one coin being tossed 7 times.

   $\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=7$.

   Let X show the number of tails. Then,

   $$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{n-r}={}^{7}C_r\cdot\left(\frac{1}{2}\right)^7.$$

   Required probability $=P(X=1)+P(X=3)+P(X=5)+P(X=7)$.

4. $P(\text{a head})=\frac{1}{2}$ and $P(\text{not a head})=\frac{1}{2}$.

   $\therefore \quad p=\frac{1}{2},\ q=\frac{1}{2}$ and $n=6$.

   $\therefore \quad P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{6}C_r\cdot\left(\frac{1}{2}\right)^6.$

   (i) $P(\text{exactly 4 heads})={}^{6}C_4\cdot\left(\frac{1}{2}\right)^6$.

   (ii) $P(\text{at least 1 head})=1-P(\text{no head})$

   $$=1-P(0\text{ head})=\left[1-{}^{6}C_0\cdot\left(\frac{1}{2}\right)^6\right].$$

   (iii) $P(\text{at the most 4 heads})=P(4\text{ or less than 4 heads})$

   $$=1-P[5\text{ heads or }6\text{ heads}]$$

   $$=1-\left[({}^{6}C_5+{}^{6}C_6)\left(\frac{1}{2}\right)^6\right].$$

5. 10 coins being tossed simultaneously is the same as one coin being tossed 10 times.

   $$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{10}C_r\cdot\left(\frac{1}{2}\right)^{10}.$$

   (i) $P(\text{exactly 3 heads})={}^{10}C_3\cdot\left(\frac{1}{2}\right)^{10}$.

(ii) P(not more than 4 heads)

$= P(X \le 4)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4)\left(\frac{1}{2}\right)^{10}$.

(iii) P(at least 4 heads)

$= P$(4 heads or 5 heads or … or 10 heads)

$= 1 - P$(0 head or 1 head or 2 heads or 3 heads)

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3)\left(\frac{1}{2}\right)^{10}$.

6. $p = \frac{3}{6} = \frac{1}{2}, q = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$ and $n = 6$.

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r} = {}^6C_5 \left(\frac{1}{2}\right)^6$.

(i) P(exactly 5 successes) $= {}^6C_5 \left(\frac{1}{2}\right)^6$.

(ii) P(at least 5 successes) $= P$[(5 successes) or (6 successes)]

$= ({}^6C_5 + {}^6C_6)\left(\frac{1}{2}\right)^6$.

(iii) P(at most 5 successes) $= P$[0 or 1 or 2 or 3 or 5 successes]

$= 1 - P$(6 successes) $= \left[1 - {}^6C_6 \cdot \left(\frac{1}{2}\right)^6\right]$.

7. $p = \frac{2}{6} = \frac{1}{3}, q = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = 4$.

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{(4-r)}$.

(i) P(exactly 3 successes) $= {}^4C_3 \cdot \left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^1$.

(ii) P(at least 2 successes) $= [P(X=2)$ or $P(X=3)$ or $P(X=4)]$

$= 1 - [P(X=0) + P(X=1)]$.

(iii) P(at most 2 successes) $= P[(X=0)$ or $(X=1)$ or $(X=2)]$.

8. $p = \frac{1}{6}$, $q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $n = 2$.

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^2C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(2-r)}$.

P[at least one 4] $= P(X=1$ or $X=2) = P(X=1) + P(X=2)$.

9. $p = \frac{6}{36} = \frac{1}{6}, q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$ and $n = 4$.

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(4-r)}$.

$\therefore \quad P(X=2) = {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2.$

10. Let $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

$\therefore \quad p = \frac{6}{36} = \frac{1}{6}, \; q = \left(1 - \frac{1}{6}\right) = \frac{5}{6}, \; n = 7.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^7C_r \cdot \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{(7-r)}$

(i) $P(X=0) = {}^7C_0 \cdot \left(\frac{5}{6}\right)^7.$

(ii) $P(X=6) = {}^7C_6 \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right).$

(iii) P(at least 6 successes) $= P$(6 successes or 7 successes)

$= P(X=6) + P(X=7)$

$= {}^7C_6 \cdot \left(\frac{1}{6}\right)^6 \cdot \left(\frac{5}{6}\right) + {}^7C_7 \cdot \left(\frac{1}{6}\right)^7.$

(iv) P(at most 6 successes) $= P(X \le 6)$

$= [1 - P(X=7)] = 1 - {}^7C_7 \cdot \left(\frac{1}{6}\right)^7.$

11. $p = \frac{6}{100} = \frac{3}{50}, \; q = \left(1 - \frac{3}{50}\right) = \frac{47}{50}$ and $n = 8.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^8C_r \cdot \left(\frac{3}{50}\right)^r \cdot \left(\frac{47}{50}\right)^{(8-r)}$

Required probability $= P$(0 defective or 1 defective)

$= P(X=0) + P(X=1)$

$= {}^8C_0 \cdot \left(\frac{47}{50}\right)^8 + {}^8C_1 \cdot \left(\frac{3}{50}\right)\left(\frac{47}{50}\right)^7.$

12. $p = \frac{6}{60} = \frac{1}{10}, q = \left(1 - \frac{1}{10}\right) = \frac{9}{10}$ and $n = 5.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r} = {}^5C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(5-r)}$

(i) P(none is defective) $= P(X=0) = {}^5C_0 \cdot \left(\frac{9}{10}\right)^5.$

(ii) P(exactly 2 are defective) $= P(X=2) = {}^5C_2 \cdot \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^3.$

13. $p = \frac{1}{20}, \; q = \left(1 - \frac{1}{20}\right) = \frac{19}{20}$ and $n = 5.$

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^5C_r \cdot \left(\frac{1}{20}\right)^r \cdot \left(\frac{19}{20}\right)^{(5-r)}$

(i) $P(X=0) = {}^5C_0 \cdot \left(\frac{19}{20}\right)^5.$

(ii) $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$

$= 1 - {}^5C_0 \cdot \left(\frac{19}{20}\right)^5$.

(iii) $P(X \leq 1) = P(X = 0) + P(X = 1)$

$= {}^5C_0 \cdot \left(\frac{19}{20}\right)^5 + {}^5C_1 \cdot \left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^4$.

14. $p = \frac{10}{100} = \frac{1}{10}$, $q = \left(1 - \frac{1}{10}\right) = \frac{9}{10}$ and $n = 6$.

$\therefore \quad P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(6-r)}$.

(i) $P(X = 2) = {}^6C_2 \cdot \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^4$.

(ii) $P(X \leq 2) = {}^6C_0 \cdot \left(\frac{9}{10}\right)^6 + {}^6C_1 \cdot \frac{1}{10} \times \left(\frac{9}{10}\right)^5 + {}^6C_2 \cdot \frac{1}{100} \times \left(\frac{9}{10}\right)^4$.

(iii) $P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2)$.

15. $p = \frac{1}{15}$, $q = \left(1 - \frac{1}{15}\right) = \frac{14}{15}$ and $n = 6$.

$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{15}\right)^r \cdot \left(\frac{14}{15}\right)^{(6-r)}$.

$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$

$= 1 - \left[{}^6C_0 \cdot \left(\frac{14}{15}\right)^6 + {}^6C_1 \cdot \left(\frac{1}{15}\right) \cdot \left(\frac{14}{15}\right)^5 + {}^6C_2 \cdot \left(\frac{1}{15}\right)^2 \cdot \left(\frac{14}{15}\right)^4\right]$

$= 1 - \left(\frac{14}{15}\right)^4 \left(\frac{59}{45}\right)$.

16. $p = \frac{1}{10}$, $q = \left(1 - \frac{1}{10}\right) = \frac{9}{10}$ and $n = 3$.

$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^3C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{(3-r)}$.

$P(X = 0) = {}^3C_0 \cdot \left(\frac{9}{10}\right)^3$.

17. $p = \frac{80}{100} = \frac{4}{5}$, $q = \left(1 - \frac{4}{5}\right) = \frac{1}{5}$ and $n = 4$.

$P(X = r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^4C_r \cdot \left(\frac{4}{5}\right)^r \cdot \left(\frac{1}{5}\right)^{(4-r)}$.

$P(X \geq 3) = P(X = 3) + P(X = 4)$

$= \left[{}^4C_3 \cdot \left(\frac{4}{5}\right)^3 \cdot \left(\frac{1}{5}\right) + {}^4C_4 \cdot \left(\frac{4}{5}\right)^4\right]$.

18. $p=\frac{1}{4}$, $q=\left(1-\frac{1}{4}\right)=\frac{3}{4}$ and $n=7$.

$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{7}C_r\cdot\left(\frac{1}{4}\right)^r\cdot\left(\frac{3}{4}\right)^{(n-r)}$

$P(X\geq 2)=1-P(X<2)$

$=1-P[(X=0)\ \text{or}\ (X=1)]$

$=1-[P(X=0)+P(X=1)]$.

19. $p=P(\text{knocking down 1 hurdle})=\left(1-\frac{5}{6}\right)=\frac{1}{6}$, $q=\frac{5}{6}$ and $n=10$.

$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{10}C_r\cdot\left(\frac{1}{6}\right)^r\cdot\left(\frac{5}{6}\right)^{(10-r)}$.

$P(X<2)=P(X=0)+P(X=1)$

$=\left[{}^{10}C_0\cdot\left(\frac{5}{6}\right)^{10}+{}^{10}C_1\cdot\left(\frac{1}{6}\right)^1\cdot\left(\frac{5}{9}\right)^9\right]$.

20. $p=\frac{1}{3}$, $q=\left(1-\frac{1}{3}\right)=\frac{2}{3}$ and $n=3$.

$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{3}C_r\cdot\left(\frac{1}{3}\right)^r\cdot\left(\frac{2}{3}\right)^{(3-r)}$.

$P(X\geq 1)=P(X=1)+P(X=2)+P(X=3)$

$=\left[{}^{3}C_1\times\frac{1}{3}\times\left(\frac{2}{3}\right)^2\right]+\left[{}^{3}C_2\times\left(\frac{1}{3}\right)^2\times\frac{2}{3}\right]+\left[{}^{3}C_3\times\left(\frac{1}{3}\right)^3\right]$

$=\left(\frac{4}{9}+\frac{2}{9}+\frac{1}{9}\right)=\frac{7}{9}$.

21. $P(X\geq 8)=P(X=8)+P(X=9)+P(X=10)$

$=\left[{}^{10}C_8\cdot\left(\frac{13}{20}\right)^8\cdot\left(\frac{7}{20}\right)^2+{}^{10}C_9\cdot\left(\frac{13}{20}\right)^9\cdot\left(\frac{7}{20}\right)+{}^{10}C_{10}\cdot\left(\frac{13}{20}\right)^{10}\right]$

$=\left[{}^{10}C_2\cdot\left(\frac{13}{20}\right)^8\cdot\frac{49}{400}+{}^{10}C_1\cdot\left(\frac{13}{20}\right)^9\cdot\frac{7}{20}+\left(\frac{13}{20}\right)^{10}\right]$.

$\therefore\quad P=\left(\frac{13}{20}\right)^8\cdot\left[\frac{441}{80}+\frac{91}{40}+\frac{169}{400}\right]=\left[\left(\frac{13}{20}\right)^8\times\frac{821}{100}\right]$.

$\Rightarrow\quad \log P=[8(\log 13-\log 20)+\log 821-\log 100]$

$=[-1.4968+2.9143-2]=-0.5825$

$=\bar{1}.4175$

$\Rightarrow\quad P=\text{antilog}\,(\bar{1}.4175)$.

22. $P(\text{white})=\frac{5}{20}=\frac{1}{4}$, $P(\text{nonwhite})=\frac{3}{4}$.

$\therefore\quad p=\frac{1}{4}$, $q=\frac{3}{4}$ and $n=4$.

$P(X=r)={}^{n}C_r\cdot p^r\cdot q^{(n-r)}={}^{4}C_r\cdot\left(\frac{1}{4}\right)^r\cdot\left(\frac{3}{4}\right)^{(4-r)}$.

(i) $P(X=0) = {}^4C_0 \cdot \left(\frac{3}{4}\right)^4$.

(ii) $P(X=4) = {}^4C_4 \cdot \left(\frac{1}{4}\right)^4$.

(iii) $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$.

23. $p = \frac{6}{10} = \frac{3}{5}, q = \left(1 - \frac{3}{5}\right) = \frac{2}{5}$ and $n = 6$.

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{3}{5}\right)^r \cdot \left(\frac{2}{5}\right)^{(6-r)}.$$

$\therefore \quad P(X=0) = {}^6C_0 \cdot \left(\frac{2}{5}\right)^6$.

24. $p = \frac{2}{6} = \frac{1}{3}, \; q = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = 3$.

$\therefore \quad \mu = np$ and $\sigma^2 = npq$.

28. $\left(np = 4 \text{ and } npq = \frac{4}{3}\right) \Rightarrow q = \frac{1}{3}$.

$\therefore \quad p = (1-q) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ and $n = \frac{4}{p} = \left(4 \times \frac{3}{2}\right) = 6$.

$\therefore \quad P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)}$

$$= {}^6C_r \cdot \left(\frac{2}{3}\right)^r \cdot \left(\frac{1}{3}\right)^{(6-r)}.$$

$P(X \geq 1) = 1 - P(X=0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{2}{3}\right)^0 \cdot \left(\frac{1}{3}\right)^6$$

$$= \left(1 - \frac{1}{3^6}\right) = \frac{728}{729}.$$

29. $np = 6$ and $npq = 2 \Rightarrow q = \frac{1}{3}$ and $p = \frac{2}{3}$.

$\left(n \times \frac{2}{3}\right) = 6 \Rightarrow n = \left(6 \times \frac{3}{2}\right) = 9$.

$\therefore \quad P(5 \text{ successes}) = {}^9C_5 \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^4$.

32. $np = 6$ and $npq = 9 \Rightarrow q = \frac{npq}{np} = \frac{9}{6} = \frac{3}{2}$.

But, q cannot be greater than 1.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If A and B are mutually exclusive events such that $P(A) = 0.4$, $P(B) = x$ and $P(A \cup B) = 0.5$, then $x = ?$

   (a) 0.2 (b) 0.1 (c) $\frac{4}{5}$ (d) none of these

2. If A and B are independent events such that $P(A) = 0.4$, $P(B) = x$ and $P(A \cup B) = 0.5$, then $x = ?$

   (a) $\frac{4}{5}$ (b) 0.1 (c) $\frac{1}{6}$ (d) none of these

3. If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B/A) = 0.4$, then $P(A/B) = ?$

   (a) 0.32 (b) 0.64 (c) 0.16 (d) 0.25

4. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, then $P(A/B) = ?$

   (a) $\frac{5}{6}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{4}{5}$

5. If A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(A' \cup B') = \frac{1}{4}$, then A and B are

   (a) independent (b) mutually exclusive

   (c) both 'a' and 'b' (d) none of these

6. It is given that the probability that A can solve a given problem is $\frac{3}{5}$ and the probability that B can solve the same problem is $\frac{2}{3}$. The probability that at least one of A and B can solve a problem is

   (a) $\frac{2}{5}$ (b) $\frac{1}{15}$ (c) $\frac{13}{15}$ (d) $\frac{2}{15}$

7. The probabilities of A, B and C of solving a problem are $\frac{1}{6}$, $\frac{1}{5}$ and $\frac{1}{3}$ respectively. What is the probability that the problem is solved?

   (a) $\frac{4}{9}$ (b) $\frac{5}{9}$ (c) $\frac{1}{3}$ (d) none of these

8. A can hit a target 4 times in 5 shots, B can hit 3 times in 4 shots, and C can hit 2 times in 3 shots. The probability that B and C hit and A does not hit is

   (a) $\frac{1}{10}$ (b) $\frac{2}{5}$

   (c) $\frac{7}{12}$ (d) none of these

9. A machine operates only when all of its three components function. The probabilities of the failures of the first, second and third component are 0.2, 0.3 and 0.5 respectively. What is the probability that the machine will fail?

(a) 0.70 (b) 0.72 (c) 0.07 (d) none of these

10. A die is rolled. If the outcome is an odd number, what is the probability that it is prime?

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{5}{12}$ (d) none of these

11. If A and B are events such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$, then $P(\overline{A} \cap B) = ?$

(a) 0.2 (b) 0.1 (c) 0.4 (d) 0.5

12. If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{5}$, then $P(\overline{B}/\overline{A}) = ?$

(a) $\frac{11}{15}$ (b) $\frac{11}{45}$ (c) $\frac{23}{60}$ (d) $\frac{37}{45}$

13. If A and B are events such that $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A/B) = ?$

(a) 0.2 (b) 0.3 (c) 0.4 (d) 0.5

14. If A and B are independent events, then $P(\overline{A}/\overline{B}) = ?$

(a) $1 - P(A)$ (b) $1 - P(B)$ (c) $1 - P(A/\overline{B})$ (d) $-P(\overline{A}/B)$

15. If A and B are two events such that $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{3}$ and $P(\overline{B}) = \frac{1}{2}$, then the events A and B are

(a) independent (b) dependent

(c) mutually exclusive (d) none of these

16. A die is thrown twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 2 has appeared at least once?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{7}$ (d) $\frac{3}{5}$

17. Two numbers are selected at random from integers 1 through 9. If the sum is even, what is the probability that both numbers are odd?

(a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{4}{9}$ (d) $\frac{5}{8}$

18. In a class, 60% of the students read mathematics, 25% biology and 15% both mathematics and biology. One student is selected at random. What is the probability that he reads mathematics, if it is known that he reads biology?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{3}{8}$ (d) $\frac{5}{8}$

19. A couple has 2 children. What is the probability that both are boys, if it is known that one of them is a boy?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

20. An unbiased die is tossed twice. What is the probability of getting a 4, 5 or 6 on the first toss and a 1, 2, 3 or 4 on the second toss?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$

21. A fair coin is tossed 6 times. What is the probability of getting at least 3 heads?

(a) $\frac{11}{16}$ (b) $\frac{21}{32}$ (c) $\frac{1}{18}$ (d) $\frac{3}{64}$

22. A coin is tossed 5 times. What is the probability that tail appears an odd number of times?

(a) $\frac{3}{5}$ (b) $\frac{2}{15}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

23. A coin is tossed 5 times. What is the probability that head appears an even number of times?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{15}$ (d) $\frac{1}{2}$

24. 8 coins are tossed simultaneously. The probability of getting 6 heads is

(a) $\frac{7}{64}$ (b) $\frac{57}{64}$ (c) $\frac{37}{256}$ (d) $\frac{249}{256}$

25. A die is thrown 5 times. If getting an odd number is a success, then what is the probability of getting at least 4 successes?

(a) $\frac{4}{5}$ (b) $\frac{7}{16}$ (c) $\frac{3}{16}$ (d) $\frac{3}{20}$

26. In 4 throws of a pair of dice, what is the probability of throwing doublets at least twice?

(a) $\frac{7}{36}$ (b) $\frac{17}{144}$ (c) $\frac{19}{144}$ (d) none of these

27. A pair of dice is thrown 7 times. If getting a total of 7 is considered a success, what is the probability of getting at most 6 successes?

(a) $\left(\frac{5}{6}\right)^7$ (b) $\left(\frac{1}{6}\right)^7$ (c) $\left(1-\frac{1}{6^7}\right)$ (d) none of these

28. The probability that a man can hit a target is $\frac{3}{4}$. He tries five times. What is the probability that he will hit the target at least 3 times?

(a) $\frac{459}{512}$ (b) $\frac{291}{364}$ (c) $\frac{371}{464}$ (d) none of these

29. The probability of the safe arrival of one ship out of 5 is $\frac{1}{5}$. What is the probability of the safe arrival of at least 3 ships?

(a) $\frac{1}{31}$ (b) $\frac{3}{52}$ (c) $\frac{181}{3125}$ (d) $\frac{184}{3125}$

30. The probability that an event E occurs in one trial is 0.4. Three independent trials of the experiment are performed. What is the probability that E occurs at least once?

(a) 0.784 (b) 0.936 (c) 0.964 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (c) **3.** (b) **4.** (d) **5.** (d) **6.** (c) **7.** (b) **8** (a) **9.** (b) **10.** (a)
11. (b) **12.** (d) **13.** (b) **14.** (a) **15.** (a) **16.** (b) **17.** (d) **18.** (b) **19.** (a) **20.** (a)
21. (b) **22.** (c) **23.** (d) **24.** (c) **25.** (c) **26.** (c) **27.** (c) **28.** (a) **29.** (c) **30.** (a)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. We have $A \cap B = \phi \Rightarrow P(A \cap B) = 0.$

$\therefore \quad P(A \cup B) = P(A) + P(B) \Rightarrow 0.5 = 0.4 + x \Rightarrow x = 0.1.$

2. We have $P(A \cap B) = P(A) \times P(B) = 0.4 \times x.$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 0.5 = 0.4 + x - 0.4x \Rightarrow 0.6x = 0.1 \Rightarrow x = \frac{1}{6}.$

3. $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}.$

$\Rightarrow \quad P(A \cap B) = P(B/A) \cdot P(A) = (0.4 \times 0.8) = 0.32.$

$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5} = 0.64.$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

$\Rightarrow \quad P(A \cap B) = \left(\frac{6}{11} + \frac{5}{11} - \frac{7}{11}\right) = \frac{4}{11}.$

$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(4/11)}{(5/11)} = \frac{4}{5} = 0.8.$

5. $P\{(A \cap B)'\} = \frac{1}{4} \Rightarrow P(A \cap B) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}.$

Since $P(A \cap B) \neq 0$, so A and B are not mutually exclusive.

$$P(A) \times P(B) = \left(\frac{1}{2} \times \frac{7}{12}\right) = \frac{7}{24} \neq P(A \cap B).$$

$\Rightarrow$ A and B are not independent.

6. Let E_1 be the event that A can solve the problem and E_2 be the event that B can solve the problem. Then, E_1 and E_2 are independent.

$\therefore \quad P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \left(\frac{3}{5} \times \frac{2}{3}\right) = \frac{2}{5}.$

$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \left(\frac{3}{5} + \frac{2}{3} - \frac{2}{5}\right) = \frac{13}{15}.$

7. $P(A') = \left(1 - \frac{1}{6}\right) = \frac{5}{6}, P(B') = \left(1 - \frac{1}{5}\right) = \frac{4}{5}$ and $P(C') = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$

P(that the problem is not solved) $= P(A'$ and B' and $C')$

$= P(A') \times P(B') \times P(C') = \left(\frac{5}{6} \times \frac{4}{5} \times \frac{2}{3}\right) = \frac{4}{9}.$

P(that the problem is solved) $= \left(1 - \frac{4}{9}\right) = \frac{5}{9}.$

8. Let E_1, E_2 and E_3 be the events that A, B and C can hit respectively.

Then, $P(E_1) = \frac{4}{5}, P(E_2) = \frac{3}{4}$ and $P(E_3) = \frac{2}{3}.$

Required probability $= P(E_2$ and E_3 but not $E_1) = P(E_2) \times P(E_3) \times P(E_1')$

$= \left\{\frac{3}{4} \times \frac{2}{3} \times \left(1 - \frac{4}{5}\right)\right\} = \left(\frac{1}{2} \times \frac{1}{5}\right) = \frac{1}{10}.$

9. Let E_1, E_2 and E_3 be the events that the 1st, 2nd and 3rd component function. Then,

$P(E_1) = (1 - 0.2) = 0.8, P(E_2) = (1 - 0.3) = 0.7$ and $P(E_3) = (1 - 0.5) = 0.5.$

P(machine fails) $= 1 - P$(machine functions)

$= 1 - P(E_1, E_2$ and $E_3) = 1 - \{P(E_1) \times P(E_2) \times P(E_3)\}$

$= 1 - (0.8 \times 0.7 \times 0.5) = (1 - 0.280) = 0.72.$

10. Here $S = \{1, 2, 3, 4, 5, 6\}.$

Let $A = \{1, 3, 5\}, B = \{2, 3, 5\}.$ Then, $A \cap B = \{3, 5\}.$

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A \cap B) = \frac{2}{6} = \frac{1}{3}.$

$\therefore \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{(1/3)}{(1/2)} = \frac{2}{3}.$

11. $B = (\overline{A} \cup A) \cap B = (\overline{A} \cap B) \cup (A \cap B)$

$\Rightarrow \quad P(B) = P(\overline{A} \cap B) + P(A \cap B)$

$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B) = (0.2 - 0.1) = 0.1.$

12. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \left(\frac{1}{4} + \frac{1}{3} - \frac{1}{5}\right) = \frac{23}{60}.$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = \left(1 - \frac{23}{60}\right) = \frac{37}{60}.$

$P(\overline{B}/\overline{A}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{A})} = \left(\frac{37}{60} \times \frac{4}{3}\right) = \frac{37}{45}.$

13. $P(B/A)=\dfrac{P(A\cap B)}{P(A)} \Rightarrow P(A\cap B)=0.6\times 0.4=0.24.$

$\therefore \quad P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.24}{0.8}=\dfrac{24}{80}=\dfrac{3}{10}=0.3.$

14. $P(\overline{A}/\overline{B})=\dfrac{P(\overline{A}\cap\overline{B})}{P(\overline{B})}=\dfrac{P(\overline{A})\cdot P(\overline{B})}{P(\overline{B})}=P(\overline{A})=1-P(A).$

15. $P(B)=1-P(\overline{B})=\left(1-\dfrac{1}{2}\right)=\dfrac{1}{2}.$

$P(A\cup B)=P(A)+P(B)-P(A\cap B) \Rightarrow P(A)=\dfrac{2}{3}$

$P(A)\cdot P(B)=\left(\dfrac{2}{3}\times\dfrac{1}{2}\right)=\dfrac{1}{3}=P(A\cap B).$

16. Let $A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

and $B=\{(1,2),(2,1),(2,2),(3,2),(2,3),(4,2),(2,4),(5,2),(2,5),(6,2),(2,6)\}$

$\therefore \quad A\cap B=\{(2,5),(5,2)\}$

$P(B/A)=\dfrac{n(A\cap B)}{n(A)}=\dfrac{2}{6}=\dfrac{1}{3}.$

17. Out of given 9 numbers, 4 are even and 5 odd.

Let $A=$ event of choosing odd number and B be the event of getting the sum even.

$n(B)=({}^4C_2+{}^5C_2)=16$ and $n(A\cap B)={}^5C_2=10.$

$\therefore \quad P(A/B)=\dfrac{n(A\cap B)}{n(B)}=\dfrac{10}{16}=\dfrac{5}{8}.$

18. Let $A=$ event of reading mathematics and B of reading biology.

$P(A)=\dfrac{60}{100}=\dfrac{3}{5}, P(B)=\dfrac{25}{100}=\dfrac{1}{4}$ and $P(A\cap B)=\dfrac{15}{100}=\dfrac{3}{20}.$

$\therefore \quad P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\left(\dfrac{3}{20}\times\dfrac{4}{1}\right)=\dfrac{3}{5}.$

19. Let $S=\{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}.$

Let $A=$ event that both are boys and $B=$ event that one of the two is a boy.

Then, $A=\{B_1B_2\}, B=\{B_1B_2, B_1G_2, G_1B_2\}$ and $A\cap B=\{B_1B_2\}.$

$\therefore \quad P(A/B)=\dfrac{n(A\cap B)}{n(B)}=\dfrac{1}{3}.$

20. Here $S=\{1,2,3,4,5,6\}$. Let $A=\{4,5,6\}$ and $B=\{1,2,3,4\}.$

$\therefore \quad P(A)=\dfrac{3}{6}=\dfrac{1}{2}$ and $P(B)=\dfrac{4}{6}=\dfrac{2}{3}.$

Clearly, A and B are independent events.

$\therefore \quad P(A\cap B)=P(A)\times P(B)=\left(\dfrac{1}{2}\times\dfrac{2}{3}\right)=\dfrac{1}{3}.$

21. In a single throw, we have $P(H)=\dfrac{1}{2}$ and $P(\text{not } H)=\dfrac{1}{2}.$

$\therefore \quad p=\dfrac{1}{2}, q=\dfrac{1}{2}$ and $n=6.$

Required probability = P(3 heads or 4 heads or 5 heads or 6 heads)

$$= {}^6C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 + {}^6C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 + {}^6C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right) + {}^6C_6 \cdot \left(\frac{1}{2}\right)^6$$

$$= \left(20 \times \frac{1}{64} + 15 \times \frac{1}{64} + 6 \times \frac{1}{64} + \frac{1}{64}\right) = \frac{42}{64} = \frac{21}{32}.$$

22. In a single throw, we have $P(T) = \frac{1}{2}$ and $P(\text{not } T) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 1)$ or $P(X = 3)$ or $P(X = 5)$

$= P(X = 1) + P(X = 3) + P(X = 5)$

$$= {}^5C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^4 + {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{5}{32} + \frac{10}{32} + \frac{1}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

23. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 0)$ or $P(X = 2)$ or $P(X = 4)$

$= P(X = 0) + P(X = 2) + P(X = 4)$

$$= {}^5C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^5 + {}^5C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1$$

$$= \left(\frac{1}{32} + \frac{10}{32} + \frac{5}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

24. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 8$.

Required probability = P(6 heads or 7 heads or 8 heads)

$= P(6 \text{ heads}) + P(7 \text{ heads}) + P(8 \text{ heads})$

$$= {}^8C_6 \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^2 + {}^8C_7 \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^1 + {}^8C_8 \cdot \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{28}{256} + \frac{8}{256} + \frac{1}{256}\right) = \frac{37}{256}.$$

25. In a single throw of a die, $P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = (1 - p) = \frac{1}{2}$ and $n = 5$.

Required probability = P(4 successes or 5 successes)

$$= {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{6}{32} = \frac{3}{16}.$$

26. In a single throw of a pair of dice, we have

$n(S) = (6 \times 6) = 36.$

Let $E = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}.$

Then, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$, $P(\text{not } E) = \left(1 - \dfrac{1}{6}\right) = \dfrac{5}{6}.$

$\therefore \quad p = \dfrac{1}{6}, q = \dfrac{5}{6}$ and $n = 4.$

Required probability = P(2 or 3 or 4 successes)

$$= {}^4C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^2 + {}^4C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right) + {}^4C_4 \cdot \left(\frac{1}{6}\right)^4$$

$$= \left(\frac{25}{216} + \frac{5}{324} + \frac{1}{1296}\right) = \frac{171}{1296} = \frac{19}{144}.$$

27. In a single throw of a pair of dice, we have

$n(S) = (6 \times 6) = 36.$

Let $E = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

$\therefore \quad p = \dfrac{6}{36} = \dfrac{1}{6}, q = \left(1 - \dfrac{1}{6}\right) = \dfrac{5}{6}$ and $n = 7.$

$P(7 \text{ successes}) = {}^7C_7 \cdot \left(\dfrac{1}{6}\right)^7 = \left(\dfrac{1}{6}\right)^7.$

$P(\text{at most 6 successes}) = 1 - P(7 \text{ successes}) = 1 - \left(\dfrac{1}{6}\right)^7 = \left(1 - \dfrac{1}{6^7}\right).$

28. Here $p = \dfrac{3}{4}, q = \left(1 - \dfrac{3}{4}\right) = \dfrac{1}{4}$ and $n = 5.$

$P(3 \text{ or } 4 \text{ or } 5 \text{ successes}) = P(3 \text{ successes}) + P(4 \text{ successes}) + P(5 \text{ successes})$

$$= {}^5C_3 \cdot \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^2 + {}^5C_4 \cdot \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} + {}^5C_5 \cdot \left(\frac{3}{4}\right)^5$$

$$= \left(\frac{135}{512} + \frac{405}{1024} + \frac{243}{1024}\right) = \frac{918}{1024} = \frac{459}{512}.$$

29. Here $p = \dfrac{1}{5}, q = \left(1 - \dfrac{1}{5}\right) = \dfrac{4}{5}$ and $n = 5.$

$\therefore \quad P(3 \text{ or } 4 \text{ or } 5 \text{ successes}) = P(3 \text{ successes}) + P(4 \text{ successes}) + P(5 \text{ successes})$

$$= {}^5C_3 \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^2 + {}^5C_4 \cdot \left(\frac{1}{5}\right)^4 \cdot \left(\frac{4}{5}\right)^1 + {}^5C_5 \cdot \left(\frac{1}{5}\right)^5$$

$$= \left(\frac{32}{625} + \frac{4}{625} + \frac{1}{3125}\right) = \frac{181}{3125}.$$

30. $p = 0.4, q = (1 - 0.4) = 0.6$ and $n = 3.$

Required probability = $P(E$ occuring at least once)

$$= {}^3C_1 \cdot (0.4)^1 \times (0.6)^2 + {}^3C_2 \cdot (0.4)^2 \times (0.6)^1 + {}^3C_3 \cdot (0.4)^3$$

$$= \left(\frac{54}{125} + \frac{36}{125} + \frac{8}{125}\right) = \frac{98}{125} = 0.784.$$